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The fundamental group and Van Kampen's theorem

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THE FUNDAMENTAL GROUP AND VAN KAMPEN'S THEOREM

A Thesis

Presented to the

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In Partial Fulfillment

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in

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Abstract

This thesis deals with the field of algebraic topology, in particular, the fundamental group of a topological space. Basic topological facts will be addressed including, but not limited to, open and closed sets, continuity, homeomorphisms, and path connectedness. The notion of homotopy classes of paths will be discussed, and the concept of homotopy will be defined. Examples and facts regarding homotopies will be addressed. The first fundamental group of a topological space will be defined, and some of its basic properties and applications will be discussed. Our mail goal is to discuss Van Kampen's Theorem in detail. It will be stated, proved, and used to compute the first fundamental group of various spaces, including the torus, the Klein bottle, and the figure eight.

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Table of Contents

List of Figures

Chapter 1

Introduction

The field of algebraic topology began taking root thousands of years ago, however, in a certain way, it was not until Alexander the Great came along that this particular field of mathematics started to become cemented into the hearts and minds of intellectuals of that era. Now, one may certainly ask how a historical figure such as Alexander the Great could possibly influence a field of mathematics? Well the answer is not that surprising. Alexander was the prized student of the great scholar and teacher Aristotle. Aristotle himself was a student of Pythagorean ideas, and so it is not surprising that he passed on some of these ideas to Alexander, namely the idea that the Earth was a sphere. Technology during that time was dim compared to modern times, so actually knowing that the Earth was spherical was more naive intuition than proven fact.

So, at Aristotle's request, Alexander took mapmakers on most of his quests and between the years of 338 and 323 BCE, Alexander the Great had conquered most of the known world [O'S07]. The actual maps that were created during the reign of Alexander the Great did not last the test of time, however the "writings of some of his generals did, and their descriptions and accounts became the basis of maps created centuries later" [O'S07].

Even still, Alexander and his generals were only able to map a proper subset of the earth. There were many regions that remained uncharted for a long period of time. Now, as we will discuss in Chapter 3, having a complete map of an object does not completely determine the shape of the object. However, if one were to have a complete and detailed map of a region, in this case the world as Alexander knew it, one could at least narrow down the possibilities. To precisely determine what kind of shape one is working with or living on, more information is needed. However, the purpose of this thesis is not to study shapes and maps. Instead, we will closely examine an object that will help us narrow down the possibilities, an object call the fundamental group. In Chapter 3, we will show that the fundamental group is indeed a group (see Theorem 3.12 for details), and that the fundamental group of a space does not depend on a choice of initial frame of reference (a basepoint). Then in Chapter 4, we will use the fundamental group and other topological ideas, such as path-connectedness, to prove Van Kampen's Theorem (see Theorem 4.6 for details), which is a theorem that allows us to compute the fundamental group of a space by considering certain open sets that are path-connected. As a result, will will then use Van Kampen's Theorem to compute the fundamental group of the sphere, the figure eight, the torus, and the Klein bottle.

Now that we have a desire to further investigate the fundamental group and Van Kampen's Theorem, the following is a more detailed outline of this thesis. In Chapter 2, we will recall many topological facts such as open and closed sets, the subspace topology, continuous functions, connected and path connected spaces, and homeomorphisms. Then in Section 2.3, we will define what it means for two functions to be homotopic, and then we will use this fact to show that homotopy is an equivalence relation (see Lemma 2.14). To end this chapter, we discuss the concept of a deformation retract and a contractible space, and then we will show that there is a special relationship between deformation retracts and homotopies (see Lemma 2.22).

In Chapter 3, we begin by discussing the history of the fundamental group and Henri Poincaré, the man responsible for bringing the fundamental group to the forefront of mainstream mathematics during the 19th century. We will then define the fundamental group as the set of all equivalence classes of loops based at x_0 and denote it as $\pi_1(X, x_0)$. We then use this definition to compute the fundamental group of uncomplicated spaces, such as $\mathbb{R}^n$ (or any convex subset of $\mathbb{R}^n$), and the one point topological space. Afterward, we progress through several lemmas to arrive at the main result for this chapter, that being the fundamental group is a group under the concatenation operation. To end this chapter, we show how a continuous map between two spaces induces a well-defined homomorphism from the fundamental group of one space to the fundamental group of the other space, and we use this construction to show that if two spaces are homotopic then their fundamental groups are isomorphic.

Finally, in Chapter 4, we begin by recalling some basic algebraic facts such as normal subgroups and free groups, which are only used in this thesis in the context of Van Kampen's Theorem and to compute the fundamental group of various topological spaces. We then use Van Kampen's Theorem to compute the fundamental group of the sphere, the figure eight, the torus, and the Klein bottle (see Section 4.3). To finish the chapter, we recall what the fundamental group and Van Kampen's Theorem have shown and outline topics for further discussion.

Chapter 2

Topology

Consider the following joke: "Three mathematicians are shown a cube and asked to describe what they see. The first, a geometer, says, 'I see a cube'. The second is a graph theorist. She ventures, 'I see eight points, connected by twelve edges'. The third, a topologist, declares, 'I see a sphere'." [Szp07] This joke, conceivably uninteresting and misunderstood by non-mathematicians, is what topology is all about. Geometers, in general, must deal at length with distances and measurements. When studying topology, on the other hand, we do not necessarily need to worry about length and measurement. Much like the joke above, "topologists are blind to angles–or the lack thereof–distances, and the exact shapes of the objects of their interest" [Szp07].

With that said, we begin our study of topology and the fundamental group by starting on a much smaller scale. We will first recall many basic prerequisites for the study of topology, namely the definition of a topology itself, the subspace topology, and continuous functions. Then, in Section 2.2, we discuss connected and path-connected spaces, and we show that the interval $[0,1]$ is connected. We also show that connectedness is a topological invariant (see Theorem 2.8) and that path-connectedness implies connectedness. We then discuss the concept of a homotopy in Section 2.3 and show that homotopy is an equivalence relation. Finally, we will explore deformation retracts and we will show that if A is a deformation retract of a space X, then A is homotopic to X (see Lemma 2.22). This is an important result, as it is needed to prove Corollary 3.21, which is used in the next chapter to compute the fundamental group of various spaces.

2.1 Topological Preliminaries and Definitions

Open and closed sets are the building blocks of topology. The following definition will begin our study.

Definition 2.1. *[Mun75] A topology on a set X is a collection Γ of subsets of X having the following properties:*

1. *The empty set and X are in Γ.*

2. *The union of the elements of any subcollection of Γ is in Γ.*

3. *The intersection of the elements of any finite subcollection of Γ is in Γ.*

We say that any subset U of X is an open set of X if U belongs to the collection Γ. A set U is closed if its compliment U^c in X is open.

Now we discuss a particular example of a topology that exists on a given set X with a particular subset $Y \subseteq X$. Since this particular topology is well known, the proof showing that this topology satisfies the three conditions in Definition 2.1 has been left out, but can be found in [Mun75].

Definition 2.2. *[Mun75] Let X be a topological space with topology Γ. If Y is a subset of X, the collection*

$$\Gamma_Y = \{Y \cap U \mid U \in \Gamma\}$$

is a topology on Y, called the subspace topology.

The subspace topology is of utmost importance for this thesis since many of the concepts we discuss deal with subsets of a topological space X. In the subspace topology there is a standard way to determine if a set is closed. We exhibit this in the following theorem, and include the proof for completeness.

Theorem 2.3. *Let Y be a subspace of a topological space X and let $A \subseteq Y$. Then A is closed in Y if and only if there exists a C closed in X with $A = C \cap Y$.*

Proof. First suppose that A is closed in Y. This implies that $Y - A$ is open in Y. Then by definition of the subspace topology, there exists a U, open in X, with $Y - A = U \cap Y$. Set $C = U^c$. Then $A = C \cap Y$.

Conversely, suppose $A = C \cap Y$ where C is closed in X. We wish to show that A is closed in Y, that is, $Y - A$ is open. We claim that $Y - A = C^c \cap Y$. We know that $Y - (C \cap Y) = (Y - C) \cup (Y - Y) = Y - C$. Therefore, we just need to show that $Y - C = (X - C) \cap Y$. So, let $y \in Y - C$. This implies that $y \in Y$ but $y \notin C$. Since $Y \subseteq X$, we have that $y \in X$, but not in C, and $y \in Y$. Therefore, $y \in (X - C) \cap Y$. Now let $x \in (X - C) \cap Y$. Then, $x \in X$, $x \notin C$, and $x \in Y$. Since $Y \subseteq X$, $x \in Y$, $x \notin C$, then $x \in Y - C$. Therefore we have that $Y - A = Y - (C \cap Y) = Y - C = (X - C) \cap Y = C^c \cap Y$. C^c is open, Y is open, therefore as the intersection of two open sets, $Y - A$ is open. Thus, A is closed. □

We now shift our attention to discussing maps between topological spaces. What will interest us most, especially in the remaining chapters, is under what conditions do certain properties of a topological space remain invariant? We begin with a definition.

Definition 2.4. *Let X and Y be topological spaces. A function $f : X \mapsto Y$ is <u>continuous</u> if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.*

In topology, and in mathematics in general, not many functions leave properties of spaces unchanged. Continuous functions are special in that they preserve some characteristics of the target space from the domain space. Therefore, for this thesis, functions appearing in mathematical statements will be assumed to be continuous, and any functions which we construct will be shown to be continuous. To this end, we will use the following lemma, known as the "Pasting Lemma", to show that the functions we construct are continuous. A proof can be found in [Mun75].

Lemma 2.5. *[Mun75] [Pasting Lemma] Let A, $B \subseteq X$ be closed, with $A \cup B = X$, and let $f : X \mapsto Y$ be a function. If $f|_A : A \mapsto Y$ is continuous and $f|_B : B \mapsto Y$ is continuous, then $f : X \mapsto Y$ is continuous.*

2.2 Connected and Path-Connected Spaces

To prove and use Van Kampen's Theorem in Chapter 4, particular spaces need to be path-connected. However, before we can define what it means to be path-connected, we must first discuss what it means to be connected. We offer a definition.

Definition 2.6. *The space X is said to be connected if there does not exist a pair of nonempty, open, disjoint subsets of X whose union is X.*

As we progress through this thesis, we will be using the interval [0,1], denoted by I, to construct continuous functions. Therefore, it is beneficial for us to show that the unit interval is an example of a connected space. We present this as a lemma.

Lemma 2.7. *The unit interval [0,1] is connected.*

Proof. Assume [0,1] is disconnected, that is, there exists an $A, B \subseteq [0,1]$ where A and B are nonempty, $A \cup B = [0,1]$, and $A \cap B = \emptyset$. Since $A \cup B = [0,1]$, we assume without loss of generality that $1 \in A$. Since A is open in [0,1], there exists an ϵ-neighborhood $U_\epsilon(1) = (1 - \epsilon, 1 + \epsilon)$ such that $U_\epsilon \cap [0,1] \subseteq A$. This implies that any element of U_ϵ is an upper bound for B. Therefore, sup $B < 1$. Set sup $B = \beta$. Since [0,1] is closed and bounded and $B \subsetneq [0,1]$, we have $\beta \in [0,1]$. Yet, since the intersection of A and B is empty, β can not be in both A and B. Thus, we look at the possibility that $\beta \in B$ and $\beta \in A$ separately and arrive at a contradiction.

Case 1: Suppose $\beta \in B$. Since sup $B = \beta$, there exists an $\tilde{\epsilon}$-neighborhood about β with $U_{\tilde{\epsilon}}(\beta) \cap [0,1] \subseteq B$. This implies that there exists an $\tilde{\epsilon}$ such that $\beta + \tilde{\epsilon} \in B$. Therefore, β is not an upper bound for B. Thus, $\beta \notin B$.

Case 2: Suppose $\beta \in A$. As before, there exists an $\hat{\epsilon}$-neighborhood about β with $U_{\hat{\epsilon}}(\beta) \cap [0,1] \subseteq A$. Consider the neighborhood $U_{\hat{\epsilon}}(\beta) \cap [0,1] \subseteq A$. We remark that $\beta \neq 0$, since in that case $A = \emptyset$. So, we have that $(\beta - \hat{\epsilon}, \beta] \cap [0,1] \subseteq A$, and since $A \cap B = \emptyset$, then $(\beta - \hat{\epsilon}, \beta] \cap B = \emptyset$. Thus we have that $b \leq \beta - \frac{\hat{\epsilon}}{2} \leq \beta$, where $b \in B$. Thus, we have found an upper bound for B, namely $\beta - \frac{\hat{\epsilon}}{2}$, which is less than β. Thus, β can not be in A.

At the beginning of this argument, it was shown that $\beta \in [0,1]$, however, we have shown that $\beta \notin A$ and $\beta \notin B$. Since we assumed that $\beta \in [0,1] = A \cup B$, our original assumption must be wrong. That is, there is no such separation for [0,1]. Thus, [0,1] must be connected. □

We now turn our attention to showing that connectedness is one of the properties of a space that remains invariant under a continuous map.

Theorem 2.8. *[Mun75] The image of a connected space under a continuous map is connected.*

Proof. We will show this by contradiction. Let $f : X \mapsto Y$ be a continuous map and let X be connected. We want to show the image space $Z = f(X)$ is connected. Since the map obtained from f by restricting its range to the space Z is also continuous, it suffices to consider the case of a surjective (onto) map $g : X \mapsto Z$. Suppose that Z is not connected, that is suppose that $Z = A \cup B$ is a separation of Z into two disjoint nonempty sets which are open in Z. Then $g^{-1}(A)$ and $g^{-1}(B)$ are disjoint sets whose union is X, and they are open in X because g is continuous and they are nonempty because g is surjective. Therefore, $g^{-1}(A)$ and $g^{-1}(B)$ form a separation of X, contradicting the fact that X is connected. Thus, Z must be connected, and our theorem is true as claimed. □

Now, that we understand what it means for a space to be connected we can define what it means for the space to be path-connected. We define paths and the notion of path-connectedness below.

Definition 2.9. *[Mun75] Given points x and y of the space X, a path in X from x to y is a continuous map $f : [0,1] \mapsto X$ such that $f(0) = x$ and $f(1) = y$. A space X is said to be path − connected if every pair of points of X can be joined by a path in X.*

In order to prove Van Kampen's Theorem, we must have that certain subsets of a topological space X are path connected. So, making sure that any two points in the space can be joined by a path will be an important assumption. In addition, knowing that a space is path-connected implies that the space is connected, as the following theorem will show.

Theorem 2.10. *If X is path-connected, then X is connected.*

Proof. We will show this by contradiction. Suppose that X is path-connected but not connected, that is suppose $X = A \cup B$ where A and B are open, nonempty, and disjoint. Let $x \in A$ and $y \in B$. Since X is path connected, there exists a path $\gamma : [0,1] \mapsto X$ such that γ is continuous, $\gamma(0) = x$, and $\gamma(1) = y$. Since γ is continuous and [0,1] is connected, then $\gamma([0,1])$ is connected by Theorem 2.8. So, $\gamma([0,1]) \subseteq A$ or B. This can not happen as $\gamma(0) \in A$ and $\gamma(1) \in B$. Thus, X is connected. □

To finish our discussion of basic topological facts, we recall the notion of a homeomorphism.

Definition 2.11. *Let X and Y be topological spaces and let $f : X \mapsto Y$ be a bijection (i.e. 1-1 and onto). Then f is called a homeomorphism if f and f^{-1} are continuous. In this case, we say X is homeomorphic to Y and denote this as $X \approx Y$.*

With regard to topological spaces, a homeomorphism is the strongest of equalities, essentially corollating open sets of one space with open sets in another space. However, for our purposes, we content ourselves with using an alternate, and weaker, form of equality, namely that of a homotopy. In fact, in the next section, we show that if two spaces are homeomorphic, then they are homotopic.

2.3 Homotopy

Thus far in our discussion of preliminary topological ideas, we have been mostly interested in spaces and functions that have very special properties. However, we would like to know how to start deforming our spaces. It seems awkward that we would want to deform a space to gather information from it, but as it will be shown, if one space can be deformed into another space that is not as complex, then useful information can be ascertained from studying the less complex space. Even still, deformations change spaces. Therefore, we would like to deform our spaces by some process which preserves interesting information the space has to offer. This section is dedicated to the concept of homotopy and this deformation process. We begin our discussion with a definition.

Definition 2.12. *If X and Y are topological spaces and $f, g : X \mapsto Y$ are continuous, then f is homotopic to g if there exists $F : X \times I \mapsto Y$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$, and F is continuous. In this situation, we will call F a homotopy and we write $f \simeq g$ (f is homotopic to g). A path homotopy is a homotopy of paths (with the same endpoints) $\alpha \simeq \alpha'$ such that $F(0, s) = \alpha(0) = \alpha'(0)$ and $F(1, s) = \alpha(1) = \alpha'(1)$.*

In reference to Definition 2.12, unless otherwise noted, all homotopies between paths will be path homotopies and $\alpha \simeq \beta$ is assumed to be "α is path homotopic to β" whenever α and β are paths.

To show a concrete example of a homotopy and how they are relevant, we provide an example.

Example 2.13. *If $X = I$, $Y = \mathbb{R}^n$, and $f, g : I \mapsto \mathbb{R}^n$ with $f(0) = g(0)$ and $f(1) = g(1)$, then $f \simeq g$.*

We just need to construct a continuous map $F : I \times I \mapsto X$ such that $F(t, 0) = f(t)$ and $F(t, 1) = g(t)$. So, define $F(t, s) = (1 - s)f(t) + sg(t)$. To check that F is a homotopy, we see that if $s = 0$, we have that $F(t, 0) = (1 - 0)f(t) + 0g(t) = f(t)$, and if $s = 1$, we have that $F(t, 1) = (1 - 1)f(t) + 1g(t) = g(t)$. In addition, $F(0, s) = (1 - s)f(0) + sg(0) = f(0)$ and $f(0) = g(0)$ and that $F(1, s) = (1 - s)f(1) + sg(1) = f(1)$. Thus, since we have constructed a suitable homotopy F, we have that $f \simeq g$. □

Another way to visualize these statements is that two paths are homotopic if the image of one can be continuously be deformed into the image of the other. In the special case that the two paths have the same starting and ending points, then this deformation can be viewed as a smooth transformation from one path (when $s = 0$) to the other path (when $s = 1$).

Now that we have an idea of what a homotopy is and what it can do, we are now able to ask more interesting questions. In particular, what properties are preserved by homotopy? We begin with a more detailed treatment of homotopy by proving that $\simeq$ is an equivalence relation.

Lemma 2.14. *The relation $\simeq$ is an equivalence relation. That is, let $f, g, h : X \mapsto Y$ be continuous. Then,*

1. *$f \simeq f$.*

2. *If $f \simeq g$, then $g \simeq f$.*

3. *If $f \simeq g$ and $g \simeq h$, then $f \simeq h$.*

Proof. First, we will show that $f \simeq f$. Define $F : X \times I \mapsto Y$ such that $F(x, s) = f(x)$. Then F is the desired homotopy.

Now, we will show that if $f \simeq g$, then $g \simeq f$. So, given that $f \simeq g$, then there exists $F : X \times I \mapsto Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. We want to show that $g \simeq f$. Define $G : X \times I \mapsto Y$ such that $G(x, s) = F(x, 1 - s)$. G is the composition of continuous functions, and is therefore continuous. We notice that when $s = 0$, $G(x, 0) = F(x, 1) = g(x)$, and when $s = 1$, $G(x, 1) = F(x, 0) = f(x)$. Therefore we can conclude that $g \simeq f$.

Finally, we will show that if $f \simeq g$ and $g \simeq h$, then $f \simeq h$. Suppose $f \simeq g$ via the homotopy F and suppose $g \simeq h$ by the homotopy G. Now, define the homotopy $H : X \times I \mapsto Y$ so that $H(x,s) = F(x,2s)$ when $s \in [0, \frac{1}{2}]$ and $H(x,s) = G(x, 2s-1)$ when $s \in [\frac{1}{2}, 1]$ for any $x \in I$. To check that H is the desired homotopy, we see that when $s = 0$, then $H(x,0) = F(x,0) = f(x)$, when $s = \frac{1}{2}$, $H(x, \frac{1}{2}) = F(x,1) = G(x,0) = g(x)$ (so that H is well-defined), and when $s = 1$, $H(x,1) = G(x,1) = h(x)$. Now we just need to check that H is continuous. However, this follows easily from Theorem 2.5, where $A = X \times [0, \frac{1}{2}]$ and $B = X \times [\frac{1}{2}, 1]$ are closed sets with $A \cup B = X \times I$. Thus, H is continuous. □

Since $\simeq$ is an equivalence relation, we may define $[f]$ as the equivalence class of all continuous maps homotopic to f, similarly in the case of paths and path homotopies. That is, $g \in [f]$ if and only if $g \simeq f$. In addition, we remark that the above proof holds for path homotopies and path homotopy classes of loops as well (paths which start and end at the same place). This notion is used when we define the fundamental group in Definition 3.2.

Functions are not the only objects that can be homotopic to each other. Topological spaces can be homotopic as well. In addition, in Lemma 2.16, we show that homotopy is a weaker version of homeomorphism.

Definition 2.15. *Topological spaces X and Y are* <u>*homotopic*</u> *if there exists $f : X \mapsto Y$ and $g : Y \mapsto X$, both continuous, such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. We then write $X \simeq Y$.*

Lemma 2.16. *If $f = g$, then $f \simeq g$. If $X \approx Y$, then $X \simeq Y$.*

Proof. If $f, g : X \mapsto Y$ and $f = g$, then this implies that all functional values of f are the same as the functional values of g. Specifically, $f(x) = g(x)$ for all $x \in X$. So, to show that $f \simeq g$, we need to construct a homotopy. Define $F : X \times I \mapsto Y$ such that $F(x,s) = f(x)$. Then $F(x,0) = f(x) = g(x)$ and $F(x,1) = f(x) = g(x)$. Thus, F is the desired homotopy and $f \simeq g$.

If $X \approx Y$, then by definition there exists an f such that $f : X \mapsto Y$ and a $g : Y \mapsto X$ such that both f and g are continuous bijections. Furthermore, we have that

$f \circ g = id_Y$ and $g \circ f = id_X$ (by definition). From above, we know that if two functions are equal, then they are homotopic. Thus, since $f \circ g = id_Y$, then $f \circ g \simeq id_Y$. A similar statement can be used to show that $g \circ f \simeq id_X$, and thus $X \simeq Y$. □

So, as we have shown in Lemma 2.16, if two functions are equal, then they are homotopic and if two topological spaces are homeomorphic, then they are homotopic. This shows that homotopy is a weaker version of equivalence compared to that of homeomorphism.

2.4 Deformation Retracts

At the beginning of Section 2.2, we first mentioned the idea of changing and deforming a space into another by some smooth process, and we went on to define what a homotopy is and how to apply it. Now we can describe in detail what we mean by a deformation. So, in this section, we will focus our attention on the concept of a deformation retract. We begin with a definition.

Definition 2.17. *Let A be a subspace of a topological space X. We say that A is a deformation retract of X if there exists a homotopy $F : X \times I \mapsto X$ such that*

1. *F is continuous.*

2. *$F(a,t) = a$ for all $t \in [0,1]$ and $a \in A$.*

3. *$F(x,0) = x$ for all $x \in X$.*

4. *$F(x,1) \in A$.*

If A is a deformation retract of X, then we write $X \searrow A$.

With regard to deformation retracts, if A happens to be the one point topological space, then we can refer to the following definition.

Definition 2.18. *A space X is said to be <u>contractible</u> if $X \searrow \{e\}$, where $\{e\}$ denotes the one point topological space.*

To better illustrate the idea of a deformation retract, we offer a lemma which will be referred to throughout the remainder of the chapter.

Lemma 2.19. $\mathbb{R}^n \searrow \{0\}$*, thus* $\mathbb{R}^n$ *is a contractible space.*

Proof. We just need to define an $F : \mathbb{R}^n \times I \mapsto \{0\}$ such that the four conditions of a deformation retract are met. So, define $F : \mathbb{R}^n \times I \mapsto \mathbb{R}^n$ such that $F(x, s) = (1-s)x$. We see F is continuous. $F(0, s) = 0$ for all t. $F(x, 0) = (1-0)x = x$ for all $x \in X$. $F(x, 1) = (1-1)x = 0 \in A$. Thus, F is the desired homotopy, $\mathbb{R}^n \searrow \{0\}$, and $\mathbb{R}^n$ is contractible. □

The homotopy mentioned in Lemma 2.19 is called a straight line homotopy since $F(x, s)$ is a straight line from x to 0 as s varies from 0 to 1. Since $\mathbb{R}^n$ is a vector space, this makes sense. Thus, in more complicated spaces, these straight line homotopies need not exist.

So, what Lemma 2.19 shows is that $\mathbb{R}^n$ can be retracted to a point via a straight line homotopy. As a matter of fact, it can be shown that certain subsets of $\mathbb{R}^n$ can be retracted to a point as well (see Theorem 2.21). We will show this by first defining what it means for a set to be convex.

Definition 2.20. *[GG99] A subset X of $\mathbb{R}^n$ is <u>convex</u> if, whenever $x, y \in X$, then the straight line interval joining x to y lies in X.*

Now, we are able to prove the statement mentioned earlier, that is, not only is $\mathbb{R}^n$ contractible, but certain subsets of $\mathbb{R}^n$ are contractible as well. We state this as a theorem.

Theorem 2.21. *Let X be a convex subset of $\mathbb{R}^n$. Then X is contractible.*

Proof. Without loss of generality, let $0 \in X$. Now, restrict the homotopy in Lemma 2.19 to X. Being the restriction of a continuous map, $F(x, t)$ is still continuous and is the desired homotopy. Thus, $X \searrow \{0\}$. Therefore since $X \searrow \{0\}$ and X is contractible. □

When we start computing the fundamental group of various topological spaces at the end of Chapter 4, Theorem 2.21 will prove to be very useful. As for now, we move on to the main result of this chapter. We show an important relationship between deformation retracts and homotopies that will be used in Chapter 3 to prove Corollary 3.21, which in turn is used to compute the examples in Chapter 4.

Lemma 2.22. *If $X \searrow A$, then $X \simeq A$.*

Proof. $X \searrow A$ implies there exists $F : X \times I \mapsto X$ with the conditions listed above. So, we need to find a g and f such that $f : X \mapsto A$ and $g : A \mapsto X$ with $f \circ g \simeq id_A$ and $g \circ f \simeq id_X$. So, define $f(x) = F(x, 1)$ and $g(a) = a$. Then, $(f \circ g)(a) = f(g(a)) = f(a) = F(a, 1) = a$, which means that $f \circ g = id_A$. By Lemma 2.16, we then have that $f \circ g \simeq id_A$. In addition, $(g \circ f)(x) = g(f(x)) = g(F(x, 1)) = F(x, 1)$. Thus, we just need to show that $F(x, 1) \simeq id_X$. So, we need to find a homotopy between $F(x, 1)$ and $id_X(x)$ such that $F(x, 1) \simeq id_X(x)$. Set $H(x, s) = F(x, s)$. H is continuous since F is continuous. $H(x, 1) = F(x, 1) = (g \circ f)(x)$ and $H(x, 0) = F(x, 0) = x = id_X$. Therefore, we have found a homotopy such that $F(x, 1) \simeq id_X$, which implies that $g \circ f \simeq id_X$. Thus, since $f \circ g \simeq id_A$ and $g \circ f \simeq id_X$, $X \simeq A$. □

To end our discussion of deformation retracts and homotopies, we offer a connection between continuous and homotopic functions, which is a generalization of Example 2.13

Lemma 2.23. *If $f, g : X \mapsto \mathbb{R}^n$ are continuous, then $f \simeq g$.*

Proof. To show that $f \simeq g$, we just need to construct a homotopy. So, define $F(s, t) = (1 - t)f(s) + tg(s)$ for all s and t, where $s \in X$ and $t \in [0, 1]$. Now, $(1 - t)f(s) + tg(s) \in \mathbb{R}^n$ for all s and t. This works when the codomain is $\mathbb{R}^n$ since the notion of addition and subtraction is well-defined in $\mathbb{R}^n$. This gives us $F(s, 0) = (1 - 0)f(s) + 0g(s) = f(s)$ and $F(s, 1) = (1 - 1)f(s) + 1g(s) = g(s)$. F is the sum of two continuous functions, therefore F is continuous as well. Thus, F is a homotopy and $f \simeq g$. □

Chapter 3

The First Fundamental Group

3.1 A Brief History

What is the shape of our universe? Actually, no one knows for sure. There are theories, but no one can say with 100% certainty what the shape of the universe is. Well, what about the shape of the Earth? According to Donal O'Shea [O'S07], by the time Columbus sailed for the New World in 1492, most people believed that the Earth was spherical in shape. People in history seemed to think that if the Earth was a sphere, many naturally occuring things could be more easily explained. O'Shea remarks that "the tides, night and day, [and] the phases of the moon... could be explained by thinking of the Earth as a sphere" [O'S07]. Even still, it was more of a wish than reality for those living at that time. In fact, we could not answer this question more precisely until the 19th century. It was not until this time that most of the world's surface had been mapped. This was when the poles were explored and the interiors of some continents examined [O'S07]. So, how would one determine the shape of the universe? One possible starting point is to draw on previous experience, as those throughout history have. We can not determine the shape of the universe until it has been mapped out more carefully. If we had a map (in this case, a 3-dimensional map) of every corner of the universe, then we might be able to decipher its shape. Even still, we might even only be able to narrow down the possibilities. So we ask ourselves a simple question: Is there another way to identify the shape of an object (earth, universe, etc.) without being given a detailed map of the object? For a simple question, the answer quite complex. It is possible to narrow down the possibilities, however to say precisely what shape it is requires more information, information not pertinent to this thesis. So instead of trying to figure out what the shape of the object is, we will try to determine the possibilities. To do that, we look at closed paths on the surface. If we can identify precisely what happens to closed paths, or loops, on a surface, then we could have a better idea as to what shape the surface is. As a matter of fact, this concept was studied by another pertinent mathematician.

Henri Poincaré was a brilliant mathematician. He studied anything that interested him, and he was interested in a lot. Around 1895, he published a paper entitled Analysis Situs, which was claimed to be the origins of topology and algebraic topology. "He introduced the fundamental group and he was able to show that any 2-dimensional surface having the same fundamental group as the 2-dimensional surface of a sphere is topologically equivalent to a sphere" [OR03]. As we will show in Chapter 3, the fundamental group is defined to be the set of all equivalence classes of loops with a fixed base point. So, as it seems, Poincaré was able to answer a very difficult question in the case of dimension 2.

So, why should a genius such as Poincaré need to be discussed for this thesis? As we have already mentioned, Poincaré invented the notion of the fundamental group and completely solved the problem in the case of dimension 2. However, he pondered another question, which took roughly 200 years and a new branch of mathematics to solve. His question, however, was made in the form of a statement and is known as the Poincaré Conjecture.

Conjecture 3.1. *[Pie06] [Poincaré Conjecture] Any closed, topological space that has a trivial fundamental group which is locally homeomorphic to $\mathbb{R}^3$ is homeomorphic to S^3.*

So how does the Poincaré Conjecture relate to the topic of this paper? Well, in order to understand the statement of the Poincaré Conjecture, one must study loops on surfaces. If we know what loops do on a surface, we can better predict what kind of object they are possibly living on. The full proof of the Poincaré Conjecture is beyond the scope of this paper. So instead, we will be content with discussing, in detail, the set of all closed loops on surfaces: an object known as the fundamental group. Knowing the fundamental group of a topological space may not always tell us its shape, but it can tell us what it is not.

Therefore, in this chapter, we will begin by defining the fundamental group of a topological space. Then we will show, in detail, that the fundamental group is a group. Along the way we will compute the fundamental group of some basic topological spaces. Then in Section 3.3, we show how the fundamental group induces homomorphisms, and then in Theorem 3.20 we show that if two spaces are homotopic, then their fundamental groups are isomorphic.

3.2 The Fundamental Group: Definitions and Properties

Now that we have a historical motivation to study the fundamental group, we will shift our focus to the fundamental group itself. This section will be dedicated to proving that the fundamental group is a group, and we will compute the fundamental group of several examples. We begin with a definition.

Definition 3.2. *Define a* loop based at $x_0 \in X$ *to be a continuous function* $f : [0,1] \mapsto X$, *with* $f(0) = f(1) = x_0$. *Let* $[f]$ *denote the homotopy class of the function* f. *Then we define*

$$\pi_1(X, x_0) = \{[f] | f \text{ is a loop in } X \text{ based at } x_0\}$$

and we call $\pi_1(X, x_0)$ *the fundamental group.*

We will prove shortly (see Theorem 3.12) that $\pi_1(X, x_0)$ is a group for any topological space X and any $x_0 \in X$. The operation on this group is given in Definition 3.6, and we will show that the following loop is the identity element of $\pi_1(X, x_0)$ (see Lemma 3.10).

Definition 3.3. *Define* $e \in \pi_1(X, x_0)$ *by* $e(t) = x_0$ *for all* t.

Informally, we can say that e is the constant loop based at x_0 and that $\pi_1(X, x_0)$ is the set of all equivalence classes of loops in X based at x_0. To offer an example, we calculate the fundamental group of $X = \mathbb{R}^n$.

Example 3.4. $\pi_1(\mathbb{R}^n, x_0) = \{[e]\}$.

Let $[f] \in \pi_1(\mathbb{R}^n, x_0)$. By Lemma 2.23, $f \simeq e$. Thus, $[f] = [e]$. So then $\pi_1(\mathbb{R}^n, x_0)$ consists of only one element, namely $[e]$. Therefore, $\pi_1(\mathbb{R}^n, x_0) = \{[e]\}$. $\square$

Note that in Example 3.4 that the space we are dealing with is $\mathbb{R}^n$. We may perform a similar computation if X is a convex subset of $\mathbb{R}^n$, although our approach will require more tools introduced in Corollary 3.22, and we delay our computation until then.

We offer another example, this time $X = \{c\}$, where $\{c\}$ denotes the one point topological space.

Example 3.5. $\pi_1(\{c\}, c) = \{[e]\}$.

Let f be a loop in $\{c\}$. Our space consists of exactly one point, so $f(t) = e(t) = c$ for all t. Since $f = e$, then $f \simeq e$ by Lemma 2.16. Thus, $\pi_1(\{c\}, c) = \{[e]\}$. □

We now begin the process to show that $\pi_1(X, x_0)$ is a group under the operation $\star$, which will be defined to be as concatenation of loops.

Definition 3.6. *If γ, τ are loops or equivalence classes, then $\gamma \star \tau$ is defined by*

$$(\gamma \star \tau) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \tau(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

We call $\star$ the <u>concatenation</u> operation, and here it is used to denote the operation on paths where $\gamma(1) = \tau(0)$.

Now that we have defined what $\star$ is, we wish to exhibit a few other properties of $\star$ in an effort to ultimately establish that $\pi_1(X, x_0)$ is a group under $\star$. So, we begin by showing that $\pi_1(X, x_0)$ is closed under the operation $\star$ and that the operation $\star$ is well-defined.

Lemma 3.7. *$\pi_1(X, x_0)$ is closed under the operation $\star$.*

Proof. Let $\gamma : [0,1] \mapsto X$ such that $\gamma(0) = \gamma(1) = x_0$ and γ continuous. Also, let $\tau : [0,1] \mapsto X$ such that $\tau(0) = \tau(1) = x_0$ and τ continuous. It is clear that $(\gamma \star \tau)(0) = (\gamma \star \tau)(1) = x_0$. So, we just need to check that since γ and τ are continuous, that $\gamma \star \tau$ is continuous as well.

So, let $h : [0,1] \mapsto X$ by

$$h(t) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \tau(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

We notice that $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are closed in $[0,1]$ and that γ, τ are continuous. Thus, by the Pasting Lemma, $h(t) = \gamma \star \tau$ is continuous. Therefore, the operation $\star$ is closed. □

Lemma 3.8. *The operation $[\gamma] \star [\tau] = [\gamma \star \tau]$ is well-defined on $\pi_1(X, x_0)$. Here, the $\star$ on the left is the new operation being defined on $\pi_1(X, x_0)$ while the $\star$ on the right is that of Definition* 3.6.

Proof. To show that $\star$ is well-defined, we need to show that if $[\gamma] = [\gamma']$ and $[\tau] = [\tau']$, then $[\gamma] \star [\tau] = [\gamma'] \star [\tau']$. Thus, we have that $[\gamma] \star [\tau] = [\gamma \star \tau]$ and $[\gamma'] \star [\tau'] = [\gamma' \star \tau']$. So, the goal is to show that $[\gamma \star \tau] = [\gamma' \star \tau']$, i.e. we need to show that $\gamma \star \tau \simeq \gamma' \star \tau'$.

So, let F be a homotopy from γ to γ' and let G be a homotopy between τ and τ'. Define $H(t, s)$ by

$$H(t,s) = \begin{cases} F(2t, s) & 0 \leq t \leq \frac{1}{2} \\ G(2t-1, s) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Notice that at $t = \frac{1}{2}$, $F(1, s) = x_0 = G(0, s)$, and H is well-defined and continuous by the Pasting Lemma. We claim that H is the required homotopy between $\gamma \star \tau$ and $\gamma' \star \tau'$. At $s = 0$, we get that

$$H(t,0) = \begin{cases} F(2t, 0) & 0 \leq t \leq \frac{1}{2} \\ G(2t-1, 0) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Notice here that $F(2t, 0) = \gamma(2t)$ and that $G(2t-1, 0) = \tau(2t-1)$. Thus, at $s = 0$, $H(t, 0)$ is just $(\gamma \star \tau)(t)$. At $s = 1$, we get that

$$H(t,1) = \begin{cases} F(2t, 1) & 0 \leq t \leq \frac{1}{2} \\ G(2t-1, 1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Notice here that $F(2t, 1) = \gamma'(2t)$ and that $G(2t-1, 1) = \tau'(2t-1)$. Thus, H is the required homotopy between $\gamma \star \tau$ and $\gamma' \star \tau'$. □

We now show that $\star$ is an associative operation on $\pi_1(X, x_0)$.

Lemma 3.9. *Let α, β, γ be paths such that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$. Then $([\alpha] \star [\beta]) \star [\gamma] \simeq [\alpha] \star ([\beta] \star [\gamma])$, i.e. $\star$ is associative in $\pi_1(X, x_0)$.*

Proof. Let α, β, and $\gamma \in \pi_1(X, x_0)$. To show that $\star$ satisfies the associative property, we need to show that $(\alpha \star \beta) \star \gamma \simeq \alpha \star (\beta \star \gamma)$. Thus, we need to construct a homotopy $F : I \times I \mapsto X$ such that $F(t,0) = (\alpha \star \beta) \star \gamma$ and $F(t,1) = \alpha \star (\beta \star \gamma)$. So, define F by

$$F(t,s) = \begin{cases} \alpha(\frac{4t}{s+1}) & 0 \le t \le \frac{s+1}{4} \\ \beta(4t - s - 1) & \frac{s+1}{4} \le t \le \frac{s+2}{4} \\ \gamma(\frac{4t-s-2}{2-s}) & \frac{s+2}{4} \le t \le 1. \end{cases}$$

We note that F is continuous by the Pasting Lemma, so we just need to check that F is well-defined. We have

$$(\alpha \star \beta) \star \gamma = \begin{cases} (\alpha \star \beta)(2t) & 0 \le t \le \frac{1}{2} \\ \gamma(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

which decomposes further to

$$(\alpha \star \beta) \star \gamma = \begin{cases} \alpha(4t) & 0 \le t \le \frac{1}{4} \\ \beta(4t-1) & \frac{1}{4} \le t \le \frac{1}{2} \\ \gamma(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

We can see that this is exactly what we would get if we let $s = 0$ in $F(t,s)$. In addition,

$$\alpha \star (\beta \star \gamma) = \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ (\beta \star \gamma)(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

decomposes to

$$\alpha \star (\beta \star \gamma) = \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \beta(4t-2) & \frac{1}{2} \le t \le \frac{3}{4} \\ \gamma(4t-3) & \frac{3}{4} \le t \le 1. \end{cases}$$

We can see that this is exactly what we would get if we let $s = 1$ in $F(t,s)$. Thus, we see that F is the desired homotopy between $(\alpha \star \beta) \star \gamma$ and $\alpha \star (\beta \star \gamma)$, therefore $\star$ satisfies the associative property. $\square$

Before proceeding to the next lemma, we remark that in Lemma 3.9, we showed that $\star$ is associative on paths. However, $\star$ is associative on loops as well. This notion is imperative as we use it to prove Theorem 3.17, and we use it in the proof of the main result, Van Kampen's Theorem.

Earlier in this section, we defined the fundamental group as the set of all equivalence classes of loops in X based at x_0, and we also defined the special loop e. We now show that e is the identity element of $\pi_1(X, x_0)$.

Lemma 3.10. *The loop $e \in \pi_1(X, x_0)$ satisfies $[e] \star [\alpha] = [e \star \alpha] = [\alpha]$. Similarly, $[\alpha] \star [e] = [\alpha \star e] = [\alpha]$.*

Proof. We know that $e(t) = x_0$ for all t. We also know that since $\alpha \in \pi_1(X, x_0)$, $\alpha : I \mapsto X$, $\alpha(0) = \alpha(1) = x_0$. So, we want to construct a $H : I \times I \mapsto X$ such that $H(t, 0) = (e \star \alpha)(t)$ and $H(t, 1) = \alpha(t)$. Define

$$H(t,s) = \begin{cases} e(2t) & 0 \leq t \leq \frac{1-s}{2} \\ \alpha(\frac{2t-1+s}{1+s}) & \frac{1-s}{2} \leq t \leq 1. \end{cases}$$

H is clearly continuous by the Pasting Lemma and it is well defined since $e(2(\frac{1-s}{2})) = x_0$ and $\alpha(\frac{2(\frac{1-s}{2})-1+s}{1+s}) = \alpha(0) = x_0$. A similar argument can be used to show that $[\alpha] \star [e] = [\alpha \star e] = [\alpha]$, and the lemma is shown. □

Lemma 3.11. *Let $[\alpha] \in \pi_1(X, x_0)$. There there exists a $[\widetilde{\alpha}] \in \pi_1(X, x_0)$ such that $[\alpha] \star [\widetilde{\alpha}] = [e]$.*

Proof. First set $\widetilde{\alpha} = \alpha(1-t)$. Then, to show that $[e] = [\widetilde{\alpha}] * [\alpha]$, we need to construct a homotopy F such that $F(t, 0) = e$ and $F(t, 1) = \widetilde{\alpha} \star \alpha$. So, define F by

$$F(t,s) = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{s}{2} \\ \alpha(s) & \frac{s}{2} \leq t \leq \frac{2-s}{2} \\ \alpha(2-2t) & \frac{2-s}{2} \leq t \leq 1. \end{cases}$$

We see immediately that when $s = 0$, $F(t, s) = \alpha(0) = x_0$ and when $s = 1$,

$$F(t,1) = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2} \\ \alpha(1) & \frac{1}{2} \leq t \leq \frac{1}{2} \\ \alpha(2-2t) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

which is equivalent to

$$F(t,1) = \begin{cases} \alpha(2t) & 0 \leq t \leq \frac{1}{2} \\ \alpha(2-2t) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Noting that F is continuous by the Pasting Lemma shows that F is the required homotopy. □

We are now able to address the highlight of this section. We present this as a theorem.

Theorem 3.12. *$\pi_1(X, x_0)$ is a group under $\star$.*

Proof. This theorem follows directly from Lemmas 3.7 - 3.11. $\square$

3.3 Induced Homomorphisms

Now that we know the fundamental group is a group, we now focus our attention to showing that the fundamental group is a topological invariant of the space X.

We can say that if $f : X \mapsto Y$ and $f(x_0) = y_0$, then given a loop $\gamma : [0,1] \mapsto X$ with $\gamma(0) = \gamma(1) = x_0$, we can construct a new loop based at y_0 in Y by

$$f(\gamma(t)) : I \xrightarrow{\gamma} X \xrightarrow{f} Y.$$

So, if $[\gamma] \in \pi_1(X, x_0)$, then we can write

$$f_*([\gamma]) = [f \circ \gamma] : \pi_1(X, x_0) \mapsto \pi_1(Y, y_0).$$

Knowing this, we can form the following definition.

Definition 3.13. *[Mun75] Let $f : X \mapsto Y$ be a continuous map where $f(x_0) = y_0$. Define*

$$f_* : \pi_1(X, x_0) \mapsto \pi_1(Y, y_0)$$

by the equation

$$f_*([\gamma]) = [f \circ \gamma]$$

where γ is a loop in X based at x_0.

We are now prepared to show that f_* as defined in Definition 3.13 is a well-defined homomorphism.

Theorem 3.14. *Let f_* be as defined in Definition 3.13. Then, f_* is a well-defined homomorphism.*

Proof. First we show f_* is well-defined. We need to show that if $[\gamma] = [\beta]$, then $f_*([\gamma]) = f_*([\beta])$, i.e. if $\gamma \simeq \beta$, then $f \circ \gamma \simeq f \circ \beta$. So, assume that $\gamma \simeq \beta$. Then, there exists an H such that $H : I \times I \mapsto X$ such that H is continuous, $H(0,t) = \gamma(t)$, and $H(1,t) = \beta(t)$. We claim that $F : I \times I \mapsto Y$ defined by $F = f \circ H : I \times I \xrightarrow{H} X \xrightarrow{f} Y$ is a homotopy between $f \circ \gamma$ and $f \circ \beta$. F is continuous since both H and f are continuous,

$F(0,t) = f(H(0,t)) = f(\gamma(t))$, and $F(1,t) = f(H(1,t)) = f(\beta(t))$. Thus, our claim is confirmed, that is, F is the desired homotopy between $f \circ \gamma$ and $f \circ \beta$. Therefore, f_* is well-defined.

Next we show f_* is a homomorphism. We are given that $f_* : \pi_1(X, x_0) \mapsto \pi_1(Y, y_0)$ and we wish to show that $f_*([\alpha] \star [\beta]) = f_*([\alpha]) \star f_*([\beta])$. Since $\star$ is a well-defined operation on loops, we know that if $\alpha, \beta \in \pi_1(X, x_0)$, then $[\alpha] \star [\beta] = [\alpha \star \beta]$. So, this gives us that $f_*([\alpha] \star [\beta]) = f_*([\alpha \star \beta]) = [f \circ (\alpha \star \beta)]$. We want to show that $[f \circ (\alpha \star \beta)]$ is homotopic to $f_*([\alpha]) \star f_*([\beta]) = [f \circ \alpha] \star [f \circ \beta] = [(f \circ \alpha) \star (f \circ \beta)]$. This is confirmed when we note that

$$(f \circ (\alpha \star \beta))(t) = \begin{cases} f(\alpha(2t)) & 0 \leq t \leq \frac{1}{2} \\ f(\beta(2t-1)) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

We can then rewrite this as

$$(f \circ \alpha) \star (f \circ \beta))(t) = \begin{cases} (f \circ \alpha)(2t) & 0 \leq t \leq \frac{1}{2} \\ (f \circ \beta)(2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Therefore, $f \circ (\alpha \star \beta) = (f \circ \alpha) \star (f \circ \beta)$ and so $f \circ (\alpha \star \beta) \simeq (f \circ \alpha) \star (f \circ \beta)$ by Lemma 2.16. Thus, f_* is a homomorphism and this completes the proof. □

As a result of Theorem 3.14, we immediately get two properties of the induced homomorphism f_*.

Corollary 3.15. $(f \circ g)_* = f_* \circ g_*$

Proof. We observe $(f \circ g)_*[\alpha] = [(f \circ g) \circ \alpha] = [f \circ (g \circ \alpha)] = f_*[g \circ \alpha] = (f_* \circ g_*)[\alpha]$. □

Corollary 3.16. *If $1_x : X \mapsto X$ by $1_x(x) = x$, then $(1_x)_*$ is the identity homomorphism*

Proof. $(1_x)_*[\alpha] = [1_x \circ \alpha] = [\alpha]$ since $1_x \circ \alpha = \alpha$. □

Induced homomorphisms are going to be very useful as they are going to help us show the existence of isomorphisms between the fundamental groups of two topological spaces in Van Kampen's Theorem.

Up to this point, we have denoted the fundamental group of a space by $\pi_1(X, x_0)$, where x_0 denotes the base point. However, it is an interesting fact that the fundamental group of a topological space does not depend on the choice of base point if X is path-connected. We show that this is the case in Theorem 3.17, and once we have proved Theorem 3.17, we will be able to unambiguously denote $\pi_1(X, x_0)$ by $\pi_1(X)$ as long as X is a path-connected space.

Theorem 3.17. *Suppose X is path connected and let $x_0, x_1 \in X$. Let γ be any path from x_0 to x_1. Then there is an isomorphism $p_\gamma : \pi_1(X, x_0) \mapsto \pi_1(X, x_1)$ defined by $p_\gamma([\alpha]) = [\gamma \star \alpha \star \overline{\gamma}]$ where $\overline{\gamma}(t) = \gamma(1-t)$.*

Proof. Let $p_\gamma([\alpha]) = [\gamma \star\ \alpha \star\ \overline{\gamma}]$ as in the theorem. We claim that p_γ is a well-defined homomorphism and that there exists a τ_γ such that $\tau_\gamma \circ p_\gamma = id_{\pi_1(X,x_0)}$ and $p_\gamma \circ \tau_\gamma = id_{\pi_1(X,x_1)}$.

We show p_γ is well-defined. Let α and β be paths in X such that $\alpha \simeq \beta$ by the homotopy F. To show that p_γ is well-defined means we need to show that $p_\gamma(\alpha) \simeq p_\gamma(\beta)$. So, we need to construct a homotopy. So, define $H(t, s)$ by

$$H(t,s) = \begin{cases} \gamma(3t) & 0 \leq t \leq \frac{1}{3} \\ F(3t-1, s) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ \overline{\gamma}(3t-2) & \frac{2}{3} \leq t \leq 1. \end{cases}$$

We note immediately that when $s = 0$, $H(t, 0)$ is $p_\gamma(\alpha)$ and when $s = 1$, $H(t, 1)$ is $p_\gamma(\beta)$. Thus, H is the required homotopy and p_γ is well-defined.

We show p_γ is a homomorphism. We want to show that $p_\gamma(\alpha\star\beta) = p_\gamma(\alpha)\star p_\gamma(\beta)$. Since $\star$ is associative on paths by Lemma 3.9, $p_\gamma(\alpha) \star p_\gamma(\beta) = [(\gamma \star \alpha \star \overline{\gamma})] \star [(\gamma \star \beta \star \overline{\gamma})] = [\gamma \star (\alpha \star \beta) \star \overline{\gamma}] = p_\gamma(\alpha \star \beta)$. Thus, p_γ is a homomorphism.

Now we define τ_γ as $\tau_\gamma = p_{\overline{\gamma}}$ which we intend to be the inverse to p_γ. Then, $(\tau_\gamma \circ p_\gamma)[(\alpha)] = \tau_\gamma([\gamma \star \alpha \star \overline{\gamma}])] = p_{\overline{\gamma}}([\gamma \star \alpha \star \overline{\gamma}]) = [\overline{\gamma} \star (\gamma \star \alpha \star \overline{\gamma}) \star \gamma] = [\alpha]$. This implies that p_γ is one to one. Next, $(p_\gamma \circ \tau_\gamma)([\beta]) = p_\gamma([\overline{\gamma} \star \beta \star \gamma]) = [\gamma \star (\overline{\gamma} \star \beta \star \gamma) \star \overline{\gamma}] = [\beta]$. This implies that p_γ is onto. Thus, since p_γ is 1-1 and onto, it must be an isomorphism and the theorem is true as claimed. □

Theorem 3.17 is a very interesting theorem because it says that in a path-connected space, the fundamental group of the space does not depend on the base point. However, what if the entire space is not path connected, rather just subsets of it? Then the fundamental group of the space will be determined by the choice of base point. The following lemma will show that if we choose a basepoint in the path-connected portion of the space, then the fundamental group of the space will be determined by the path-component the basepoint is in.

Lemma 3.18. *$\pi_1(X, x_0) \cong \pi_1(X_0, x_0)$ where X_0 is the path component containing x_0.*

Proof. Define the inclusion map i by $i : X_0 \mapsto X$. Then $i_* : \pi_1(X_0, x_0) \mapsto \pi_1(X, x_0)$ (see Definition 3.13). In addition, define $p : \pi_1(X, x_0) \mapsto \pi_1(X_0, x_0)$ by $p([\alpha]) = [\alpha]$. Note that $im([\alpha]) \subseteq X_0$ since I is path connected, and the continuous image of a connected space is connected. Now, it is clear that $i_* \circ p = id$ and $p \circ i_* = id$, so $\pi_1(X, x_0) \cong \pi_1(X_0, x_0)$. □

We now offer a relationship between homotopies and induced homomorphisms that will aide us in proving Theorem 3.20. Lemma 3.19 is a technical result designed to keep track of basepoints with regard to induced homomorphisms.

Lemma 3.19. *If $f \simeq g$, then $g_*(\alpha) = \overline{\gamma} \star f_*(\alpha) \star \gamma$ for some path γ taking $g(x_0)$ to $f(x_0)$.*

Proof. We know that $f : (X, x_0) \mapsto (Y, y_0)$ which by definition implies that $f_* : \pi_1(X, x_0) \mapsto \pi_1(Y, f(x_0))$. Similarly, $g_* : \pi_1(X, x_0) \mapsto \pi_1(Y, g(x_0))$. In addition, $f \simeq g$, which implies that there exists an $F : X \times I \mapsto X$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. So, define paths γ and $\overline{\gamma}$ by $\gamma(t) = F(t, x_0)$ and $\overline{\gamma}(t) = \gamma(1 - t)$, and define $H(t, s)$ by

$$H(t,s) = \begin{cases} F(3ts, x_0) & 0 \le t \le \frac{1}{3} \\ F(s, \alpha(3t-1)) & \frac{1}{3} \le t \le \frac{2}{3} \\ F((3-3t)s, x_0) & \frac{2}{3} \le t \le 1. \end{cases}$$

Then we see immediately that $H(t, 0) = e \star g(\alpha) \star e$, which is homotopic to $g_*(\alpha)$, and

$$H(t,1) = \begin{cases} F(3t, x_0) & 0 \le t \le \frac{1}{3} \\ F(1, \alpha(3t-1)) & \frac{1}{3} \le t \le \frac{2}{3} \\ F(3-3t, x_0) & \frac{2}{3} \le t \le 1, \end{cases}$$

which is equal to $\overline{\gamma} \star f(\alpha) \star \gamma$. Once we note that H is continuous by the Pasting Lemma, we see that H is the desired homotopy. Thus, we have shown that $g_*(\alpha) = \overline{\gamma} \star f_*(\alpha) \star \gamma$ $\in \pi_1(Y, g(x_0))$, and the lemma is shown. □

Now that we fully understand homotopies and induced homomorphisms, we can now use these facts to show that if two spaces are homotopic, their fundamental groups are isomorphic. We present this as a theorem.

Theorem 3.20. *If $X \simeq Y$ and X and Y are path-connected, then $\pi_1(X) \cong \pi_1(Y)$.*

Proof. Since $X \simeq Y$, then by Definition 2.15, there exists a $f : X \mapsto Y$ and a $g : Y \mapsto X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. So then by Lemma 3.19, for some γ, we have $(g \circ f)_*(\alpha) = (g_* \circ f_*)(\alpha) = \overline{\gamma}_* \star (id_X)_*(\alpha) \star \gamma_* = \overline{\gamma} \star \alpha \star \gamma = p_\gamma(\alpha)$ where p_γ is the isomorphism in Theorem 3.17 (note $\overline{\overline{\gamma}} = \gamma$). This implies that $g_* \circ f_*$ is an isomorphism as well. Thus, $g_* \circ f_*$ is 1-1, which means that f_* must be 1-1. Similarly, $(f_* \circ g_*)(\alpha) = \overline{\eta} \star (id_Y)_*(\alpha) \star \eta = \overline{\eta} \star \alpha \star \eta = p_\eta(\alpha)$. This implies that f_* is onto. Thus, since we have shown that f_* is 1-1 and onto, $f*$ must be an isomorphism showing that $\pi_1(X, x_0) \cong \pi_1(Y, f(x_0))$. By Theorem 3.17, $\pi_1(X) \cong \pi_1(X, x_0)$, similarly for $\pi_1(Y) \cong \pi_1(Y, f(x_0))$. So, we have that $\pi_1(X) \cong \pi_1(X, x_o) \overset{f_*}{\cong} \pi_1(Y, f(x_0)) \cong \pi_1(Y)$. □

As a direct result of Theorem 3.20, we can now prove several corollaries, including Corollary 3.22, which was first mentioned after Example 3.4, and we now have the appropriate tools to prove.

Corollary 3.21. *If $X \searrow A$, then $\pi_1(X) \cong \pi_1(A)$.*

Proof By Lemma 2.22, if $X \searrow A$, then $X \simeq A$. By Theorem 3.20, this gives us that $\pi_1(X) \cong \pi_1(A)$. □

Corollary 3.22. *If X is a convex subset of $\mathbb{R}^n$, then $\pi_1(X) = \{[e]\}$.*

Proof. This is immediate when we note Theorem 3.20, Example 3.4, and Corollary 3.21. □

Corollary 3.23. *Let C denote a contractible space and let $\{c\}$ denote the one point topological space. Then, $\pi_1(C) \cong \pi_1(\{c\}) \cong \{[e]\}$.*

Proof. Suppose C is contractible. Thus by Definition 2.18, $C \searrow \{c\}$, which implies that $C \simeq \{c\}$ by Lemma 2.22. Thus, by Theorem 3.20, $\pi_1(C) \cong \pi_1(\{c\})$. Therefore, since $\pi_1(\{c\}) = \{e\}$ by Example 3.5, we get that $\pi_1(C) \cong \pi_1(\{c\}) \cong \{e\}$. □

Corollary 3.21 is an important result of this chapter, as it assists us in our computations in Chapter 4. Informally, it says that if a one space deformation retracts to another space, then the fundamental groups of the two spaces are isomorphic. This is why at the beginning of Section 2.4 we wanted to have some process that allowed us to deform one space into another space. We no longer have to compute the fundamental group of a space if we know that it deformation retracts to a less complicated space since their fundamental groups are isomorphic by Corollary 3.21. The examples we offer at the end of Chapter 4 will also show the usefulness of Corollary 3.21 in conjunction with Van Kampen's Theorem. In the meantime, we offer an example that utilizes the techniques established thus far.

Example 3.24. $\pi_1(\mathbb{R}^n) \cong \pi_1(\{0\}) = \{[e]\}$.

By Example 2.19, $\mathbb{R}^n \searrow \{0\}$. Corollary 3.21 then gives us that $\pi_1(\mathbb{R}^n) \cong \pi_1(\{0\})$. Then, by Example 3.4, $\pi_1(\mathbb{R}^n) = \{e\}$. Thus, we get that $\pi_1(\mathbb{R}^n) \cong \pi_1(\{0\}) = \{e\}$. $\square$

Before concluding this chapter, we offer one final referenced result ([Hat02]). It will prove to be highly useful in computing some of the examples at the end of Chapter 4. We omit the proof as it uses techniques not pertinent to this thesis. We begin by defining S^1.

Definition 3.25. *Let $\mathbb{C}$ be the complex numbers. Define the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, where $|z|$ is the modulus of z.*

Example 3.26. *[Hat02]*

If we let $a(t) = e^{2\pi\sqrt{-1}t}$, then $\pi_1(S^1) \cong F(a)$ where $F(a)$ denotes the free group on one generator (see Section 4.1 for a discussion of free groups). Here, $a(t)$ refers to the movement along the unit circle counterclockwise. $\square$

Chapter 4

Van Kampen's Theorem

In Chapter 3, we defined the fundamental group of a topological space. We then used this definition to compute the fundamental group of a few basic topological spaces. These spaces were chosen because the manner in which we computed the fundamental group was direct and did not require many prerequisites. However, we now wish to compute the fundamental group of more complex spaces, namely the spheres S^n ($n \geq 2$), the figure eight, the torus, and the Klein bottle. In general, it is not an easy task to compute the fundamental group of a space. However, we can examine the characteristics of these four spaces in order to use Van Kampen's Theorem to compute the fundamental group of each.

Therefore, in this chapter, we will recall some basic algebraic facts that will be useful in proving Van Kampen's Theorem. We will then state, prove, and use Van Kampen's Theorem in order to compute the fundamental groups of the spheres S^n ($n \geq 2$), the figure eight, the torus, and the Klein bottle. Finally, we will end the chapter with a brief summary of what we have done in this thesis and what questions are still left to be answered.

4.1 Algebraic Facts

Let us begin by recalling what it means for a subgroup to be normal.

Definition 4.1. *If G is a group and N is a subgroup of G, then N is* normal *in G (or N is a normal subgroup of G) if $gN = Ng$ for all $g \in N$ and we denote this by $N \lhd G$.*

Now, if $N \lhd G$, then there exists the quotient group G/N where

$$G/N = \{gN | \text{distinct left cosets of } N \text{ in } G\}.$$

It is a well known fact that G/N is a group with the well-defined operation given by

$$(xN)(yN) = (xy)N.$$

Also recall that if G is an abelian group, i.e. a group whose operation is commutative, then all subgroups of G are normal.

Now, one might wonder if a set X is given, is there any way to turn X (possibly adding elements to X) into a group? There is, and the operation needed to make it a group is as expected since a given set X may lack the appropriate structure of a binary operation. We demonstrate this construction where X is a set with either one or two elements in Definition 4.2. There is a more general construction for other sets, but we will not need this more general construction in this thesis. Following the definition we give some brief remarks.

Definition 4.2. *Let $X = \{a, b\}$ be a set. Then we define $F(a, b)$ as*

$$F(a, b) = \{a^{i_1}b^{j_1}...a^{i_k}b^{j_k} | i_s, j_s \in \mathbb{Z}\}.$$

We say $F(a, b)$ is "the free group on two generators" and it is a group with operation juxtaposition (described below). In addition, $a^{-1} \in F(a, b)$ is the formal inverse of a (analogously for b), and $a^n \in F(a, b)$ is repeated products of a or a^{-1} (if $n < 0$). In addition, the empty word functions as the identity e, and $a^0 = b^0 = e$.

The set $F(X)$ is a group using the operation of juxtaposition, which is as follows: let $a^{i_1}b^{j_1}....a^{i_n}b^{j_n} \in F(X)$ and let $a^{p_1}b^{q_1}....a^{p_s}b^{q_s} \in F(X)$. Then $a^{i_1}b^{j_1}....a^{i_n}b^{j_n}a^{p_1}b^{q_1}....a^{p_s}b^{q_s} \in F(X)$ as well (see [Mun75]). Furthermore, when examining Definition 4.2 more closely, we can make the following observations:

1. These words may be unreduced in that $a^1b^0a^3 = a^4$, for example.

2. a and b may be replaced with groups G and H to form $G * H$ (the free product of G and H), where words with letters coming from the same group are multiplied within that group. We use this approach in the statement and proof of Van Kampen's Theorem.

3. If $X = \{a\}$ (a set of one element), then $F(a, b)$ reduces to $F(a)$ by equating b (and its powers) with the identity e.

We give a brief example of a familiar free group that we have already seen from Chapter 3.

Example 4.3. *This example studies the free group on the set $X = \{a\}$.*

Since $a \in X$, we need to include $a^{-1}, a^2, e, ...,$ etc. So,

$$F(a) = \{e, a, a^{-1}, a^2, a^{-2}, ...\}$$

We can see that $F(a) \cong \mathbb{Z}$ via the isomorphism $a^k \mapsto k$. Thus, the free group generated by a one element set is isomorphic to the set of integers. It is easy to see that the free group on one generator is the only abelian nontrivial free group on a set X. □

Now that we understand the free group on one generator, we recall Example 3.26 from the end of Chapter 3. In this example we showed that $\pi_1(S^1) \cong F(a)$. We can now state the following corollary.

Corollary 4.4. $\pi_1(S^1) \cong \mathbb{Z}$.

Proof: By Example 3.26, we know that $\pi_1(S^1) \cong F(a)$. Then, Example 4.3 gives us that $F(a) \cong \mathbb{Z}$. Therefore, by transitivity, we have that $\pi_1(S^1) \cong \mathbb{Z}$. □

We end this section with a useful fact about free products that will help us prove Van Kampen's Theorem in the next section. A proof can be found in [Mun75].

Lemma 4.5. *Let $G, H,$ and N be groups. In addition, let $\lambda : G \mapsto N$ and $\mu : M \mapsto N$ be homomorphisms. Then there exists $\Phi : G * H \mapsto N$ such that $\Phi|_G = \lambda$ and $\Phi|_H = \mu$.*

4.2 Van Kampen's Theorem

In 1924, Egbert Van Kampen graduated high school with reports of a "remarkable mathematical talent" ([OR04]). He then went to college at the University of Leiden, and in 1931 he traveled to the United States to accept a position at Johns Hopkins University to teach and research mathematics. It was at this time that Van Kampen began his intricate work in the field of topology, and not long after beginning his professorship at Johns Hopkins, Van Kampen developed what we now know as Van Kampen's Theorem. Van Kampen's Theorem is a theorem that enables us to compute the fundamental group of a topological space X by instead considering the fundamental groups of subsets of X (see Theorem 4.6 for specific details). led to the United States to accept a position at Johns Hopkins University to teach and research mathematics. It was at this time that Van Kampen began his intricate work in the field of topology, and not long after beginning his professorship at Johns Hopkins, Van Kampen developed what we now know as Van Kampen's Theorem. Van Kampen's Theorem is a theorem that enables us to compute the fundamental group of a topological space X by instead considering the fundamental groups of subsets of X (see Theorem 4.6 for specific details).

Before proceeding with the theorem, let us first consider the following observation where $A_1, A_2 \subseteq X$, a topological space, and the labeled maps are inclusions.

$$(A_1 \cap A_2) \xrightarrow{i_1} A_1 \xrightarrow{j_1} (A_1 \cup A_2)$$
$$(A_1 \cap A_2) \xrightarrow{i_2} A_2 \xrightarrow{j_2} (A_1 \cup A_2)$$

Notice that since i_1, i_2, j_1 and j_2 are inclusions, by Theorem 3.14, $j_1 i_1(a) = j_2 i_2(a)$ for all $a \in A_1 \cap A_2$. Thus, there exists induced homomorphisms

$$\begin{aligned}
(i_1)_* &: \pi_1(A_1 \cap A_2) \mapsto \pi_1(A_1), \\
(i_2)_* &: \pi_1(A_1 \cap A_2) \mapsto \pi_1(A_2), \\
(j_1)_* &: \pi_1(A_1) \mapsto \pi_1(A_1 \cup A_2), \\
(j_2)_* &: \pi_1(A_2) \mapsto \pi_1(A_1 \cup A_2), \\
(j_1 \circ i_1)_* = (j_2 \circ i_2)_* &: \pi_1(A_1 \cap A_2) \mapsto \pi_1(A_1 \cup A_2).
\end{aligned}$$

In order to prove Van Kampen's Theorem, we will need to show that two groups ($\pi_1(X, x_0)$ and Q, a particular group defined in Theorem 4.6) are isomorphic. In order to do this, we will actually construct a map and show it is an isomorphism. We will begin by using the inclusion maps i_1, i_2, j_1, and j_2 and the diagram above to construct a map Φ, as in Lemma 4.5. We will show that Φ is onto. Then we will show that Φ induces a homomorphism, Ψ, that we will show is an isomorphism showing Q and $\pi_1(X, x_0)$ are isomorphic.

We now state Van Kampen's Theorem, and prove it according to the outline mentioned above.

Theorem 4.6. *[Van Kampen's Theorem] Let $X = A_1 \cup A_2$, where A_1, A_2, and $A_1 \cap A_2$ are path-connected, and let $x_0 \in A_1 \cap A_2$. Set $Q = \frac{\pi_1(A_1)*\pi_1(A_2)}{N}$. Then*

$$\pi_1(X, x_0) \cong Q,$$

where N is the smallest normal subgroup generated by the elements $[(j_1 i_1)_*(w)]*[(j_2 i_2)_*(w)]^{-1}$ for $w \in \pi_1(A_1 \cap A_2)$. We denote N by $<< [(j_1 i_1)_*(w)] * [(j_2 i_2)_*(w)]^{-1} \mid w \in \pi_1(A_1 \cap A_2) >>$.

Proof: First, if we let $[\alpha] \in \pi_1(A_1)$, then $(i_2)_*[\alpha] \in \pi_1(A_1 \cup A_2)$. Similarly, if $[\alpha] \in \pi_1(A_2)$, then $(j_2)_*[\alpha] \in \pi_1(A_1 \cup A_2)$. So, by Lemma 4.5, we can define the map Φ such that

$$\Phi : \pi_1(A_1) * \pi_1(A_2) \mapsto \pi_1(A_1 \cup A_2)$$

where $\Phi \mid_{\pi_1(A_1)} = (j_1)_*$ and $\Phi \mid_{\pi_1(A_2)} = (j_2)_*$. We claim that Φ is onto. Before showing that Φ is onto, we note that if $[\gamma] \in \pi_1(A_1 \cup A_2)$, then we may define a <u>*factorization*</u> of $[\gamma]$ as an expression where

$$[\gamma] = [\alpha_1] * ...... * [\alpha_k]$$

where each $\alpha_i \in \pi_1(A_1)$ or $\pi_1(A_2)$, and for $p = 1$ or 2, we abuse notation to express $(j_p)_*(\alpha_i) = \alpha_i$ (the abuse is that we are writing $\alpha_i \in \pi_1(A_p)$ and $\alpha_i = (j_p)_*(\alpha_i) \in \pi_1(X)$). To show that Φ is onto, we need to show that any element $[\gamma] \in \pi_1(A_1 \cup A_2)$ has a factorization. So, let $\gamma : [0,1] \mapsto A_1 \cup A_2$. Before we can show that $[\gamma]$ has a factorization, we need to construct a partition of $[0,1]$ so that if $0 = s_0 < s_1 < ... < s_n = 1$ is the partition such that $\gamma_i = \gamma|_{[s_i, s_{i+1}]}$, then $im(\gamma_i) \subseteq A_1$ or A_2. This claim follows from a straightforward compactness argument that we outline here. Since $\gamma \in \pi_1(A_1 \cup A_2)$, $\gamma^{-1}(A_p) = \cup[(a_i^{(p)}, b_i^{(p)}) \cap [0,1]]$ is an open cover of the compact set $[0,1]$. So, there exists a finite subcover $(a_i^{(p)}, b_j^{(p)})$. Let $S = \{a_i^{(p)}, b_i^{(p)}\} \cup \{0\}$, and let $s_0 = 0 \in S$. Then, since S is finite, we may choose the next largest $s_1 \in S$ so that $s_1 > s_0$. Now, $im(\gamma|_{[0,s_1)}) \subseteq A_1$ or A_2 by construction. Suppose $im(\gamma|_{[0,s_1)}) \subseteq A_1$, but $\gamma(s_1) \in A_2 \setminus A_1$. $\gamma^{-1}(A_2)$ is open, $s_1 \in \gamma^{-1}(A_2)$, so there exists a neighborhood $(s_1 - \epsilon, s_1 + \epsilon) \subseteq \gamma^{-1}(A_2)$.

Choose $s_{\frac{1}{2}} = s_1 - \frac{\tilde{\epsilon}}{2}$, where $\tilde{\epsilon} = \epsilon$ if $(s_1 - \epsilon) > 0$ and $\tilde{\epsilon} = \frac{s_1}{2}$ if $s_1 - \epsilon \leq 0$. Now $im(\gamma|_{[s_0, s_{\frac{1}{2}}]}) \subseteq A_1$ and $im(\gamma|_{[s_{\frac{1}{2}}, s_1]}) \subseteq A_2$. Let $s_2 > s_1$ be the next greatest element of S. Consider $\gamma_1 = \gamma|_{[s_1, s_2]}$. We know that $[s_1, s_2) \subseteq (a_i^p, b_i^p)$ for some i, so $im(\gamma|_{[s_1, s_2)}) \subseteq A_2$ since $\gamma(s_1) \in A_2$. If $\gamma(s_2) \in A_1 \setminus A_2$, perform the adjustment above so that we refine the partition $s_1 < s_{\frac{3}{2}} < s_2$. Repeat at most $n < \infty$ times to produce a partition as needed. Now, this partition is not necessarily a factorization for $[\gamma]$ since the image of each $\gamma(s_i)$ is not necessarily a loop in X. So, we now use this partition to construct a proper factorization.

So, let $[\gamma] \in \pi_1(A_1 \cup A_2)$, and let $0 = s_0 < s_1 < ... < s_n = 1$ be a partition of $[0,1]$ (as outlined above). Then, since A_1, A_2, and $A_1 \cap A_2$ are path-connected, define $\eta_i : \gamma(s_i) \mapsto x_0$ and $\overline{\eta_i} : x_0 \mapsto \gamma(s_i)$, defined by $\overline{\eta_i}(t) = \eta_i(1-t)$, as paths from $\gamma(s_i)$ to x_0 and vice versa, respectively, so that $im(\eta_i) \subseteq A_1, A_2$, or $A_1 \cap A_2$. Therefore,

$$\begin{aligned}[\gamma] &= [\gamma_1 * \eta_1 * \overline{\eta_1} * \gamma_2 * \eta_2 * \overline{\eta_2} * ... * \overline{\eta}_{k-1} * \gamma_k] \\ &= [\gamma_1 * \eta_1] * [\overline{\eta_1} * \gamma_2 * \eta_2] * ... * [\overline{\eta}_{k-1} * \gamma_k].\end{aligned}$$

Thus, we may define $[\alpha_i] \in \pi_1(A_1)$ or $\pi_1(A_2)$ by

$$\begin{aligned}\alpha_1 &= \gamma_1 * \eta_1 \\ \alpha_2 &= \overline{\eta_1} * \gamma_2 * \eta_2 \\ &\vdots \\ \alpha_k &= \overline{\eta}_{k-1} * \gamma_k,\end{aligned}$$

which gives a factorization for $[\gamma]$ as $[\gamma] = [\alpha_1] * ...... * [\alpha_k]$. Therefore, $[\gamma] \in \pi_1(A_1 \cup A_2)$ is factorable (up to homotopy), thus Φ is onto.

It is possible that there exist multiple factorizations for a given $[\gamma]$, so we do not expect Φ to be an isomorphism. For example, what if $[\gamma] = [\alpha_1] * ..... * [\alpha_k] = [\beta_1] * ..... * [\beta_l]$. For the purposes of this proof, we will say that these factorizations are *equivalent* if one can get from one factorization to the other via finite combinations of the following (or their inverses): (1) replace $[\alpha_i] * [\alpha_{i+1}]$ with $[\alpha_i * \alpha_{i+1}]$ when both $\alpha_i, \alpha_{i+1} \in \pi_1(A_1)$ or $\pi_1(A_2)$, or (2) if $\alpha \in \pi_1(A_1 \cap A_2)$, then view $\alpha \in \pi_1(A_1)$ instead of $\pi_1(A_2)$, or vice versa. Thus, equivalent factorizations of $[\gamma] \in \pi_1(A_1 \cup A_2)$ correspond to identical elements of Q since property (1) replaces elements which correspond to the same element in the free product (thus the same element in Q), and property (2) shifts representatives for the same equivalence classes modulo N (thus producing the same element in Q).

We study to what extent Φ is not 1-1 and claim that Φ induces another map Ψ where

$$\Psi : Q \mapsto \pi_1(A_1 \cup A_2), \text{ defined by } \Psi(gN) = \Phi(g).$$

We claim that Ψ is onto, linear, well-defined, and 1-1, thus Ψ is an isomorphism needed to show that $\pi_1(A_1 \cup A_2)$ is isomorphic to Q.

Step 1. We first show that the map Ψ defined above is onto. So, let $[f] \in \pi_1(A_1 \cup A_2)$. Then, since Φ is onto, there exists a g such that $\Phi(g) = [f]$. But, $\Psi(gN) = \Phi(g) = [f]$. So, Ψ is onto as well.

Step 2. Next we show that the map Ψ is linear (i.e. a homomorphism). Let $g, h \in \pi_1(A_1) * \pi_1(A_2)$. Then $\Psi((gN)(hN)) = \Psi(ghN) = \Phi(gh) = \Phi(g)\Phi(h) = \Psi(gN)\Psi(hN)$. Essentially, Ψ is a homomorphism since Φ is a homomorphism.

Step 3. Now we show that the map Ψ is well-defined. Suppose that $gN = hN$. This implies that $h^{-1}gN = N$. Since Ψ is linear, $\Psi(hN)^{-1}\Psi(gN) = \Psi(h^{-1}N)\Psi(gN) = \Psi(h^{-1}gN) = \Psi(N)$. Since Φ is linear, and since the maps i_p and j_p are inclusions, Φ sends any element of the form $[(j_1i_1)_*(w)] * [(j_2i_2)_*(w)]^{-1}$ to e, then Φ sends any element of N (generated by these elements) to the identity. So, $\Psi(N) = \Phi(n) = e$ for any $n \in N$. Thus, $\Psi(gN) = \Psi(hN)$ and Ψ is well-defined.

Step 4. We are now left to show that Ψ is 1-1. Suppose that $\Psi(f_1 * \ldots\ldots * f_k) = \Psi(g_1 * \ldots\ldots * g_l) = [f] \in \pi_1(A_1 \cup A_2)$. We prove Ψ is 1-1 by showing that any two factorizations of f are equivalent. So, If we can show that $f_1 * \ldots\ldots * f_k$ and $g_1 * \ldots\ldots * g_l$ are equivalent, then they define the same element in Q, hence Ψ would be 1-1.

We know that $f \simeq f_1 * \ldots\ldots * f_k$ and $f \simeq g_1 * \ldots\ldots * g_l$. Thus, since $\simeq$ is an equivalence relation, we have that $f_1 * \ldots\ldots * f_k \simeq g_1 * \ldots\ldots * g_l$. So, there exists a homotopy F between them.

So, to begin, we use a partitioning strategy similar to earlier to partition $I \times I$ into rectangles (as shown in Figure 4.1) such that F maps each rectangle into entirely A_1 or A_2. We observe that the factorizations for f may have a different number of components (corresponding to the top and bottom of Figure 4.1). However, drawing all possible horizontal and vertical lines in $I \times I$ and refining the partition ensures that we are able to partition $I \times I$ as Figure 4.1 suggests. In addition, we label each square (starting from the lower left hand corner and wrapping one line up back to the left upon reaching the right edge of $I \times I$) as $R_1, R_2, ..., R_n$ and we denote by R_r one of these squares. Also, since f is a loop based at x_0, we label the sides of Figure 4.1 as x_0.

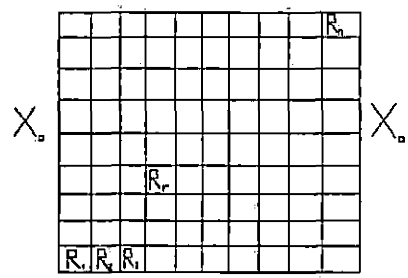

Figure 4.1: Decomposition of $I \times I$ into squares

We are now going to show that the two factorizations of $[f]$ are equivalent. To do this, let us consider the image (under F) of a particular horizontal trio of adjacent rectangles from Figure 4.1.

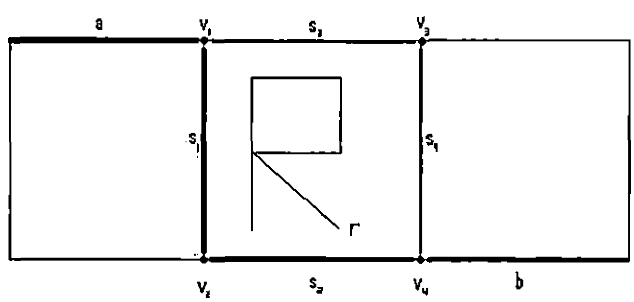

Figure 4.2: Trio of rectangles from $I \times I$ mapped into X

We define γ_r to be the image under F of the loop that starts and ends at x_0 in X by proceeding through $I \times I$ in a particular way that we describe presently. Observing Figure 4.2 may seem awkward since the trio of rectangles is in $I \times I$ and not X, but we can still use this diagram to describe a loop in X as long as we note that what we really want is the image (under F) of each part of the diagram. In addition, to simplify the notation, we are going to write $F(a)$ as a, $F(v_1)$ as v_1, etc. This will enable us to work with what we can actually see in $I \times I$, and not its image in X.

So, let γ_r be the image under F of the loop that starts at x_0, and then traverses a path by first following a to v_1, s_1 to v_2, s_2 to v_4, and then back to x_0 by b (again note that this is really taking place in X and not $I \times I$). In Figure 4.2, the dark black line signifies $F|_{\gamma_r}$. Then γ_{r+1} is the loop that starts at x_0, and then traverses a path in X via F by first following a to v_1, s_2 to v_3, s_4 to v_4, and then back to x_0 by b. By labeling γ_r and γ_{r+1} as such, we are able to equate the factorization $g_1 * \ldots\ldots * g_l$ with γ_{n+1} and the factorization $f_1 * \ldots\ldots * f_k$ with γ_0. The goal now is to show that for all $r = 0, ..., n$ that γ_r is equivalent to γ_{r+1}. ling γ_r and γ_{r+1} as such, we are able to equate the factorization $g_1 * \ldots\ldots * g_l$ with γ_{n+1} and the factorization $f_1 * \ldots\ldots * f_k$ with γ_0. The goal now is to show that for all $r = 0, ..., n$ that γ_r is equivalent to γ_{r+1}.

Since A_1, A_2, and $A_1 \cap A_2$ are path-connected, we may define $\delta_{v_i}(t) : v_i \mapsto x_0$ as a path in either A_1 or A_2 starting at v_i and ending at x_0, then $\overline{\delta_{v_i}}(t) = \delta_{v_i}(1-t) : x_0 \mapsto v_i$ is a path starting at x_0 and ending at v_i. Then a factorization for γ_r in Figure 4.2 would be

$$[(a * \delta_{v_1})] * [\overline{\delta}_{v_1} * s_1 * s_2 * \delta_{v_4}] * [(\overline{\delta}_{v_4} * b)]$$

and a factorization for γ_{r+1} in Figure 4.2 would be

$$[(a * \delta_{v_1})] * [\overline{\delta}_{v_1} * s_3 * s_4 * \delta_{v_4}] * [(\overline{\delta}_{v_4} * b)]$$

Observe that the factorizations exhibited by γ_r and γ_{r+1} are nearly identical, with the exception being $s_1 * s_2$ and $s_3 * s_4$. So, the problem is solved if we can show that $s_1 * s_2 \simeq s_3 * s_4$. We will show this by first considering $I \times I$ labeled as in Figure 4.3. Then,

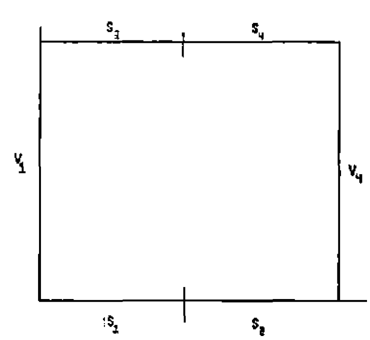

Figure 4.3: Labeling of $I \times I$ to show $s_1 * s_2 \simeq s_3 * s_4$

through a series of continuous operations, we will show how we can embed Figure 4.3 into $I \times I$ so that we may map it, by F, into X, enabling us to show that $s_1 * s_2$ and $s_3 * s_4$ are homotopic.

Step 1. Embed the $I \times I$ in Figure 4.3 in $\mathbb{R}^2$. Then, scale and translate it so that it is centered at the origin and each side has a length of two (see Figure 4.4). Scaling and translating in $\mathbb{R}^2$ is a continuous operation.

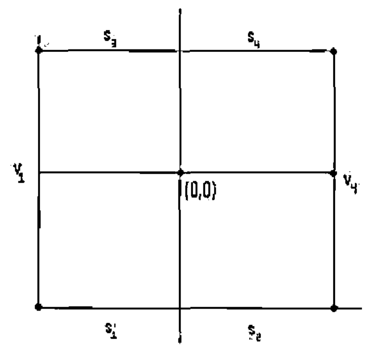

Figure 4.4: $I \times I$, after being rescaled and recentered

Step 2. Now, we wish to turn our recentered square into a hexagon by some continuous fashion. Define the maps Ω_+ and Ω_- by

$$\Omega_+ : (x, y) \mapsto (x, (2 - x)y)$$
$$\Omega_- : (x, y) \mapsto (x, (2 + x)y).$$

Then, operate on the square in Figure 4.4 by using the following rules:

$$\Omega_1(x, y) = \begin{cases} \Omega_+(x, y) & y \geq 0 \\ (x, y) & y \leq 0 \end{cases}$$

$$\Omega_2(x, y) = \begin{cases} \Omega_-(x, y) & y \leq 0 \\ (x, y) & y \geq 0. \end{cases}$$

$\Omega_2 \cup \Omega_1$ is continuous by the Pasting Lemma. The resulting figure can be seen in Figure 4.5. We label pieces of this hexagon for future reference.

Step 3. We now wish to transform the hexagon in Figure 4.5 into a square by reducing segments v_1 and v_4 to a point on the x-axis. So, operate on the hexagon by the following rule:

$$C(x, y) = \begin{cases} (x, y - 1) & (x, y) \in P_2 \\ (x, 0) & (x, y) \in P_3 \\ (x, y + 1) & (x, y) \in P_1. \end{cases}$$

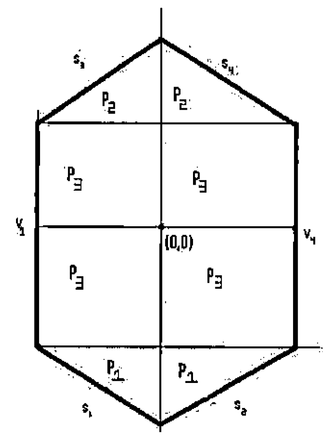

Figure 4.5: Resulting Hexagon, after shifting the top and bottom of $I \times I$

The function $C(x, y)$ is continuous by the Pasting Lemma. The resulting figure can be seen in Figure 4.6.

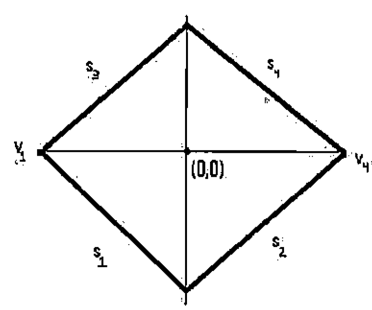

Figure 4.6: Square, which is similar to R_r

Step 4. If we examine Figure 4.6 more closely, it looks very similar to the R_r square from Figures 4.1 and 4.2, with edges of s_1, s_2, s_3, and s_4. In addition, by construction, every square in Figure 4.1 is mapped into X by the the homotopy F, which includes the R_r square. So, let us scale and rotate (both continuous operations) Figure 4.6 so that it resembles the R_r square. The final figure is given in Figure 4.7. Now all we need to do is take Figure 4.7, embed it into $I \times I$ (as a subset), and then map it into X by F.

Looking back, the goal of this entire process was to show that Ψ is 1-1. We wanted to prove this by showing that any two factorizations of γ are equivalent. We then

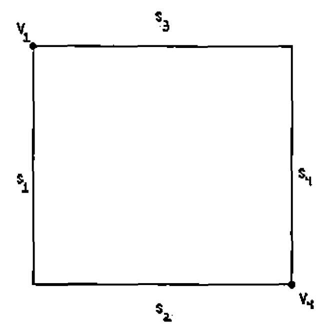

Figure 4.7: Final square used to show $s_1 * s_2 \simeq s_3 * s_4$

found that this problem was solved if we could show that $s_1 * s_2 \simeq s_3 * s_4$. What we have done in steps 1 through 4 is exactly that. In each step, we either used or constructed a continuous map that suited our purposes. In doing so, we have constructed a homotopy showing that $s_1 * s_2 \simeq s_3 * s_4$. Since $s_1 * s_2 \simeq s_3 * s_4$, we then have that the paths γ_r and γ_{r+1} in Figure 4.2 are equivalent. However, this process could be repeated for *any* horizontal trio of adjacent squares with appropriately trivial adjustments to the left or right of $I \times I$. Thus, we have that γ_r is equivalent to γ_{r+1} for all $r = 0, ..., n$. This implies that $f_1 * ...... * f_k = \gamma_0$ is equivalent to γ_1, γ_1 is equivalent to γ_2, ... , γ_{n-1} is equivalent to γ_n, and γ_n is equivalent to $\gamma_{n+1} = g_1 * ...... * g_l$. Therefore, $f_1 * ...... * f_k$ is equivalent to $g_1 * ...... * g_l$ thus making Ψ 1-1, and the theorem is shown. □

4.3 Examples

In this section we aim to use Van Kampen's Theorem in computing the fundamental group of various topological spaces, namely the sphere, the figure eight, the torus, and the Klein bottle. We offer these as examples.

Example 4.7. *Compute $\pi_1(S^n)$ (the n-dimensional sphere), for $n \geq 2$.*

Let x_0 be a point on the equator of the sphere. By Theorem 3.17, up to isomorphism, it does not matter which point on the equator we choose. In addition, let $A_1 = S^n$-{south pole} and let $A_2 = S^n$-{north pole}. Then $A_1 \cap A_2 = S^n$ - {north and south pole} and A_1, A_2, and $A_1 \cap A_2$ are path-connected. We see that $A_1 \searrow \{x_0\}$, $A_2 \searrow \{x_0\}$, and $A_1 \cap A_2 \searrow$ {equator of S^n} and the equator of S^n is homeomorphic to S^{n-1} by the map $(x_1, ..., x_n, 0) \mapsto (x_1, ..., x_n)$. Thus, By Corollary 3.21, we have that $\pi_1(A_1) = \{e\}$, $\pi_1(A_2) = \{e\}$, $\pi_1(A_1 \cap A_2) = \mathbb{Z}$ (by Example 3.26). Therefore by Van Kampen's Theorem we have that

$$\pi_1(S^n) = \frac{\pi_1(A_1) * \pi_1(A_2)}{N} = \frac{\{e\} * \{e\}}{N} = < e > .$$

□

Example 4.8. *Denote the figure 8 in $\mathbb{R}^2$ as $S^1 \vee S^1$. Compute $\pi_1(S^1 \vee S^1)$.*

The figure eight can be seen in Figure 4.8. It is the result of taking the one point union of two circles, in this case at point x. What we want to do to compute the fundamental group is find 2 open subsets, A_1 and A_2, of the figure eight whose union is the figure eight, and that A_1, A_2, and $A_1 \cap A_2$ are path-connected.

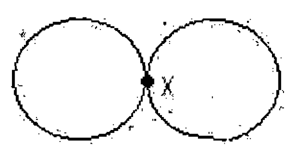

Figure 4.8: The Figure Eight

So, we let A_1 be as in Figure 4.9

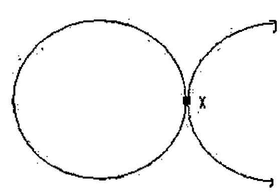

Figure 4.9: A_1 for the Figure Eight

and we let A_2 be as in Figure 4.10.

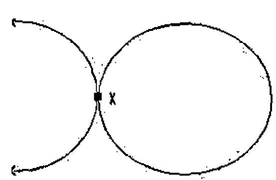

Figure 4.10: A_2 for the Figure Eight

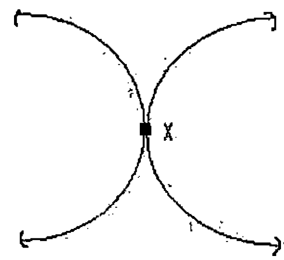

Figure 4.11: $A_1 \cap A_2$ for the Figure Eight

The subset $A_1 \cap A_2$ can be seen in Figure 4.11.

Thus we see that $A_1 \searrow S^1, A_2 \searrow S^1$, and $A_1 \cap A_2 \searrow \{x\}$. So by Corollary 3.21 and Example 3.26, $\pi_1(A_1) \cong F(a)$ where a is a generator of $\pi_1(S^1)$ in Example 3.26, $\pi_1(A_2) \cong F(b)$ where b is a generator of $\pi_1(S^1)$ in Example 3.26, and $\pi_1(A_1 \cap A_2) = \{e\}$. In addition since the fundamental group of the intersection is generated by the identity element, $N = \{e\}$. Therefore by Van Kampen's Theorem we have that

$$\pi_1(S^1 \vee S^1) = \frac{\pi_1(A_1) * \pi_1(A_2)}{N} = \frac{F(a) * F(b)}{\{e\}} = F(a) * F(b) = F(a, b).$$

□

Example 4.9. *Compute* $\pi_1(\mathbb{T}^2)$*, where* $\mathbb{T}^2 = S^1 \times S^1$ *(the torus).*

The torus can be seen by the following diagram where opposite sides are identified with the same orientation.

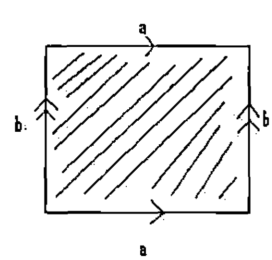

Figure 4.12: The Torus

So we let A_1 be as in Figure 4.13 where the region inside the dotted circle has been deleted. Then we let A_2 be as in Figure 4.14 where everything except the frame is included. This gives us $A_1 \cap A_2$, which can be see in Figure 4.15.

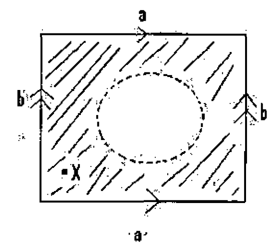

Figure 4.13: A_1 for the Torus

Thus, we see that $A_1 \searrow \{\text{wire frame}\} \cong \mathbb{Z} * \mathbb{Z} = F(a,b)$ via Example 4.8 and $A_2 \searrow \{x\}$. In addition, $A_1 \cap A_2 \searrow S^1$. Now, we just need to compute N. Let $< w > \in \pi_1(A_1 \cap A_2)$. Then, $(j_2)_*(j_1)_*(w) = e$ and $(i_2)_*(i_1)_*(w) = aba^{-1}b^{-1}$ in $\pi_1(A_1 \cup A_2)$. Therefore $N = << aba^{-1}b^{-1} >>$. More specifically,

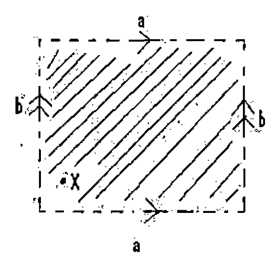

Figure 4.14: A_2 for the Torus

$aba^{-1}b^{-1} = e$ implies that $ab = ba$. Therefore, by Van Kampen's Theorem we have that

$$\pi_1(\text{torus}) = \frac{\pi_1(A_1) * \pi_1(A_2)}{N} = \frac{F(a,b) * \{e\}}{<< aba^{-1}b^{-1} >>} \cong \frac{F(a,b)}{<< ab = ba >>} \cong \mathbb{Z} \oplus \mathbb{Z}.$$

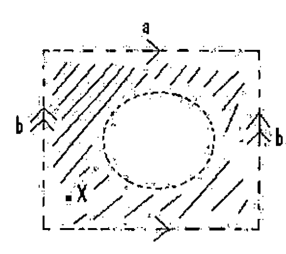

Figure 4.15: $A_1 \cap A_2$ for the Torus

□

Example 4.10. *Denote the Klein bottle by K. Compute $\pi_1(K)$.*

The Klein bottle is defined as a quotient space, where opposite sides in Figure 4.16 are glued as shown (the left and right sides are glued with the same orientation, while the top and bottom are glued with opposite orientations).

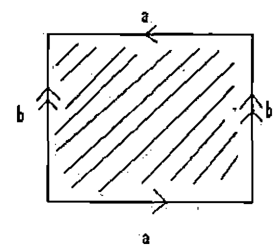

Figure 4.16: The Klein Bottle

Similarly to the last example, we let A_1 be represented by Figure 4.17

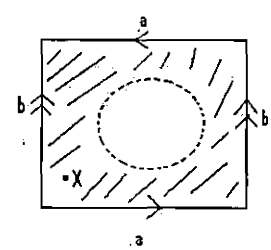

Figure 4.17: A_1 for the Klein Bottle

and we let A_2 be represented by Figure 4.18.

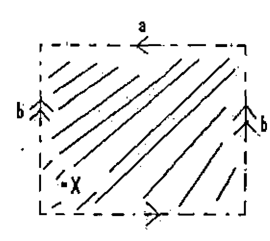

Figure 4.18: A_2 for the Klein Bottle

Then $A_1 \cap A_2$ is given by Figure 4.19.

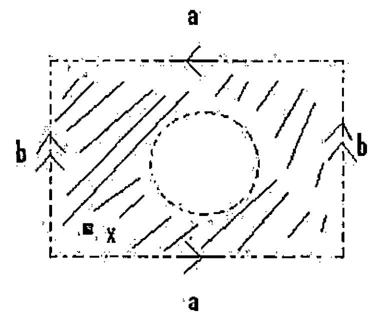

Figure 4.19: $A_1 \cap A_2$ for the Klein Bottle

We see that $A_1 \searrow \{\text{wire frame}\} \cong \mathbb{Z} * \mathbb{Z} = F(a, b)$ and $A_2 \searrow \{x\} \cong \{e\}$. In addition, $A_1 \cap A_2 \searrow S^1$. Now, we just need to compute N. Let $< w > \in \pi_1(A_1 \cap A_2)$. Then, $j_1(w) = e$ since $j_{1*} : \pi_1(A_1 \cap A_2) \mapsto \pi_1(A_2) = e$. This implies that $(j_2 j_1)_*(w) = e \in \pi_1(A_1 \cup A_2)$. So, quotienting out by N equates $(j_2 j_1)_*(w)$ with e in $\pi_1(A_1 \cup A_2)$. In addition, $(i_2)_*(i_1)_*(w) = abab^{-1}$ in $\pi_1(A_1 \cup A_2)$. Therefore, by Van Kampen's Theorem we have that

$$\pi_1(K) = \frac{\pi_1(A_1) * \pi_1(A_2)}{N} = \frac{F(a, b) * \{e\}}{<< abab^{-1} >>} \cong \frac{F(a, b)}{<< aba = b >>}.$$

There is not a generally accepted way to express this group in any other more recognizable way than as a quotient of generators of a free group by some relations. □

4.4 Conclusion

The motivation behind this entire thesis was to develop some way of computing the fundamental group of a topological space. From the start, we knew that there would be no feasible way to compute the fundamental group of every topological space. We just wanted to compute the fundamental group of a large class of topological spaces; spaces that met certain requirements. This led us to Van Kampen's Theorem. This theorem will allow us to compute the fundamental group of any topological space as long as certain subsets of the space are path-connected. Thus, before we could discuss Van Kampen's Theorem, we had to study path-connectedness and homotopies (as homotopies deal with the fundamental group itself). In addition, because Van Kampen's Theorem uses the notion of deformation retracts in its computations, we had to investigate the concept of a deformation retract. So, in order for us to proceed with proving and using Van Kampen's Theorem, we had to actually start with the most basic of topological ideas, that being "what is an open set?"

Now, even though we discussed many concepts and ideas in this thesis, we had to do it in order to fully understand each piece of the larger picture. Each topic in this thesis, from open sets to algebraic facts, is used in some form in proving Van Kampen's Theorem. Yet, since we highlighted and discussed many details, hopefully the reader understands how useful this particular theorem can be, and how it stands as a wonderful bridge between algebra and topology.

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